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THE EXISTENCE OF WARPING FUNCTIONS ON RIEMANNIAN WARPED PRODUCT MANIFOLDS

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Abstract. In this paper, when N is a compact Riemannian manifold of class (A), we consider the existence of some warping functions on Riemannian warped product manifolds $M = [a, \infty) \times_f N$ with prescribed scalar curvatures.

1. Introduction

One of the basic problems in the differential geometry is to study the set of curvature functions over a given manifold.

The well-known problem in differential geometry is whether a given metric on a compact Riemannian manifold is necessarily pointwise conformal to some metric with constant scalar curvature or not.

In a recent study ([5]), Jung and Kim have studied the problem of scalar curvature functions on Lorentzian warped product manifolds and obtaind partial results about the existence and nonexistence of Lorentzian warped metric with some prescribed scalar curvature function.

In this paper, we study also the existence and nonexistence of Riemannian warped metric with prescribed scalar curvature functions on some Riemannian warped product manifolds.

By the results of Kazdan and Warner ([6, 7, 8]), if N is a compact Riemannian $n-$manifold without boundary $n \geq 3$, then N belongs to one of the following three catagories:

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

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(B) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [6], [7] and [8], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

In [9] and [10], the author considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric.

In this paper, when N is a compact Riemannian manifold, we consider the existence of warping functions on a warped product manifold $M = [a, \infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. That is, it is shown that if the fiber manifold N belongs to class (A) then M admits a Riemannian metric with some negative scalar curvature near the end outside a compact set.

2. Main results

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, \infty) \to R^+$ be a smooth function, where a is a positive number. A Riemannian warped product of N and $[a, \infty)$ with warping function f is defined to be the product manifold $([a, \infty) \times_f N, g^{'})$ with

$$g^{'} = dt^2 + f^2(t)g. \tag{2.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t, x)$ of $g^{'}$ is given by the equation

$$R(t, x) = \frac{1}{f^2(t)}[R(g)(x) - 2nf(t)f^{''}(t) - n(n-1)|f^{'}(t)|^2] \tag{2.2}$$

for $t \in [a, \infty)$ and $x \in N$ (For details, [1] or [3]).

If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (2.2) can be changed into

$$\frac{4n}{n+1} u^{''}(t) + R(t,x)u(t) - R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian $n-$manifold without boundary.

If N admits a Riemannian metric of negative or zero scalar curvature, then we let $u(t) = t^{\alpha}$ in (2.3), where $\alpha > 1$ is a constant. We have

$$R(t,x) \le \frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} < 0, \quad t > \alpha.$$

Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [3], we have the following theorem.

Theorem 2.1. *For $n \ge 3$, let $M = [a, \infty) \times_f N$ be the Riemannian warped product $(n+1)$-manifold with N compact n-manifold. Suppose that N is in class (A) or (B), then on M there is a Riemannian metric of negative scalar curvature outside a compact set.*

Proposition 2.2. *Suppose that $R(g) = -\dfrac{4n}{n+1}k^2$ and $R(t,x) = R(t) \in C^{\infty}([a,\infty))$. Assume that for $t > t_0$, there exist an upper solution $u_+(t)$ and a lower solution $u_-(t)$ such that $0 < u_-(t) \le u_+(t)$. Then there exists a solution $u(t)$ of equation (2.3) such that for $t > t_0, 0 < u_-(t) \le u(t) \le u_+(t)$.*

Proof. We have only to show that there exist an upper solution $\tilde{u}_+(t)$ and a lower solution $\tilde{u}_-(t)$ such that for all $t \in [a, \infty)$, $\tilde{u}_-(t) \le \tilde{u}_+(t)$. Since $R(t) \in C^{\infty}([a,\infty))$, there exists a positive constant b such that $|R(t)| \le \dfrac{4n}{n+1}b^2$ for $t \in [a, t_0]$. Since

$$\begin{aligned} &\frac{4n}{n+1}u_+''(t) + R(t)u_+(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} \\ &\le \frac{4n}{n+1}(u_+''(t) + b^2 u_+(t) + k^2 u_+(t)^{1-\frac{4}{n+1}}), \end{aligned}$$

if we divide the given interval $[a, t_0]$ into small intervals $\{I_i\}_{i=1}^n$, then for each interval I_i we have an upper solution $u_{i+}(t)$ by parallel transporting $\cos Bt$ such that $0 < \frac{1}{\sqrt{2}} \leq u_{i+}(t) \leq 1$. That is to say, for each interval I_i,

$$\begin{aligned}
&\frac{4n}{n+1}u''_{i+}(t) + R(t)u_{i+}(t) + \frac{4n}{n+1}k^2 u_{i+}(t)^{1-\frac{4}{n+1}} \\
&\leq \frac{4n}{n+1}(u''_{i+}(t) + b^2 u_{i+}(t) + k^2 u_{i+}(t)^{1-\frac{4}{n+1}}) \\
&= \frac{4n}{n+1}(-B^2 \cos Bt + b^2 \cos Bt + k^2 (\cos Bt)^{1-\frac{4}{n+1}}) \\
&= \frac{4n}{n+1}\cos Bt(-B^2 + b^2 + k^2 (\cos Bt)^{-\frac{4}{n+1}}) \\
&\leq \frac{4n}{n+1}\cos Bt(-B^2 + b^2 + k^2 2^{\frac{2}{n+1}}) \\
&\leq 0
\end{aligned}$$

for large B, which means that $u_{i+}(t)$ is an (weak) upper solution for each interval I_i. Then put $\tilde{u}_+(t) = u_{i+}(t)$ for $t \in I_i$ and $\tilde{u}_+(t) = u_+(t)$ for $t > t_0$, which is our desired (weak) upper solution such that $\frac{1}{\sqrt{2}} \leq \tilde{u}_+(t) \leq 1$ for all $t \in [a, t_0]$. Put $\tilde{u}_-(t) = \frac{1}{\sqrt{2}}e^{-\alpha t}$ for $t \in [a, t_0]$ and some large positive α, which will be determined later, and $\tilde{u}_-(t) = u_-(t)$ for $t > t_0$. Then, for $t \in [a, t_0]$,

$$\begin{aligned}
&\frac{4n}{n+1}u''_{i-}(t) + R(t)u_{i-}(t) + \frac{4n}{n+1}k^2 u_{i-}(t)^{1-\frac{4}{n+1}} \\
&\geq \frac{4n}{n+1}(u''_{i-}(t) - b^2 u_{i-}(t)) \\
&= \frac{4n}{n+1}\frac{1}{\sqrt{2}}e^{-\alpha t}(\alpha^2 - b^2) \\
&\geq 0
\end{aligned}$$

for large α. Thus $\tilde{u}_-(t)$ is our desired (weak) lower solution such that for all $t \in [a, \infty)$, $0 < \tilde{u}_-(t) \leq \tilde{u}_+(t)$. $\square$

In [1], the authors consider the nonexistence of warping functions on Riemannian warped product manifolds $M = [a, \infty) \times_f N$ when N belongs to class (A) with $R(g) = -\frac{4n}{n+1}k^2$.

PROPOSITION 2.3. *Suppose that N belongs to class (A). Let g be a Riemannian metric on N of dimension $n(\geq 3)$. We assume that $R(g) = -\frac{4n}{n+1}k^2$, where k is a positive constant. On $M = [a, \infty) \times_f N$, there does not exist a Riemannian warped product metric*

$$g^{'} = dt^2 + f^2(t)g$$

with scalar curvature

$$R(t) \geq -\frac{n(n-1)}{t^2}$$

for all $x \in N$ and $t > t_0 > a$, where t_0 and a are positive constants.

However, in this paper, when N is a compact Riemannian manifold of class (A), we consider the existence of some warping functions on Riemannian warped product manifolds $M = [a, \infty) \times_f N$ with prescribed scalar curvatures. If $R(t, x)$ is also the function of only t-variable, then we have the following theorems.

THEOREM 2.4. *Suppose that $R(g) = -\frac{4n}{n+1}k^2$. Assume that $R(t, x) = R(t) \in C^\infty([a, \infty))$ is a positive function such that*

$$-bt^s \leq R(t) \leq -\frac{4n}{n+1}\frac{C}{t^\alpha}, \quad for \quad t \geq t_0,$$

where $t_0 > a, \alpha < 2, C$ and b are positive constants, and s is a positive integer. Then equation (2.3) has a positive solution on $[a, \infty)$.

Proof. We let $u_+(t) = t^m$, where m is some positive number. Then we have

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} + R(t)u_+(t) \\
&\leq \frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1}\frac{C}{t^\alpha}u_+(t) \\
&= \frac{4n}{n+1}t^m[\frac{m(m-1)}{t^2} + \frac{k^2}{t^{\frac{4}{n+1}m}} - \frac{C}{t^\alpha}] \\
&\leq 0, \quad t \geq t_0,
\end{aligned}$$

for some large t_0, which is possible for large fixed m since $\alpha < 2$. Hence, $u_+(t)$ is an upper solution. Now put $u_-(t) = e^{-\beta t}$, where β is a positive constant. Then

$$\begin{aligned}
&\frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_-(t)^{1-\frac{4}{n+1}} + R(t)u_-(t) \\
&\geq \frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_-(t)^{1-\frac{4}{n+1}} - bt^s u_-(t) \\
&= e^{-\beta t}[\frac{4n}{n+1}\beta^2 + \frac{4n}{n+1}k^2 e^{\beta t[\frac{4}{n+1}]} - bt^s] \\
&\geq 0, \quad t \geq t_0
\end{aligned}$$

for some large t_0, which means that $u_-(t)$ is a lower solution. And we can take β so large that $0 < u_-(t) < u_+(t)$. So by Proposition 2.2, we obtain a positive solution. $\square$

The above theorem implies that if $R(t)$ is not rapidly decreasing and less than some negative function, then equation (2.3) has a positive solution.

THEOREM 2.5. *Suppose that* $R(g) = -\dfrac{4n}{n+1}k^2$. *Assume that* $R(t,x) = R(t) \in C^\infty([a,\infty))$ *is a negative function such that*

$$-bt^s \leq R(t) \leq -\frac{C}{t^2}, \quad for \quad t \geq t_0$$

where $t_0 > a, b$ *and* C *are positive constants, and* s *is a positive integer. If* $C > n(n-1)$, *then equation (2.3) has a positive solution on* $[a,\infty)$.

Proof. In case that $C > n(n-1)$, we may take $u_+(t) = C_+ t^{\frac{n+1}{2}}$, where C_+ is a positive constant. Then

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} + R(t)u_+(t) \\
&\leq C_+\frac{4n}{n+1}t^{\frac{n-3}{2}}[\frac{n^2-1}{4} + k^2 C_+^{-\frac{4}{n+1}} - \frac{n+1}{4n}C] \\
&\leq 0,
\end{aligned}$$

which is possible if we take C_+ to be large enough since $\dfrac{(n+1)(n-1)}{4} - \dfrac{n+1}{4n}C < 0$. Thus $u_+(t)$ is an upper solution. And we take $u_-(t)$ as in Theorem 2.4. In this case, we also obtain a positive solution. $\square$

REMARK 2.6. *The results in Theorem 2.4, and Theorem 2.5 are almost sharp as we can get as close to* $-\dfrac{n(n-1)}{t^2}$ *as possible. For example, let* $R(g) = -\dfrac{4n}{n+1}k^2$ *and* $f(t) = t\ln t$ *for* $t > a$. *Then we have*

$$R = -\frac{1}{t^2}\left[\frac{4n}{n+1}\frac{k^2}{(\ln t)^2} + \frac{2n}{\ln t} + n(n-1)(1 + \frac{1}{\ln t})^2\right],$$

which converges to $-\dfrac{n(n-1)}{t^2}$ *as* t *goes to* ∞.

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